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Strings in Noncommutative Spacetime

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Abstract: Free bosonic strings in noncommutative spacetime are investigated. The string spectrum is obtained in terms of light-cone quantization. We construct two different models. In the first model the critical dimension is still required to be 26 while only extreme high energy spectrum is modified by noncommutative effect. In the second model, however, the critical dimension is reduced to be less than 26 while low-energy (massless) spectrum only contains degrees of freedom of our four dimensional physics.

Keywords: Bosonic Strings, Non-Commutative Geometry.

Contents

1. Introduction

There is a long-standing belief that spacetime must change its nature at the distance comparable to the Planck scale in any quantum theories including gravity. A natural generalization of quantum mechanism principle to quantum gravity is to introduce uncertainty principle among spacetime coordinates which prevents one from measuring positions to better accuracies than the Planck length[1]: The momentum and energy required to perform a measurement at these scales will modify the geometry of spacetime. The simplest and naive realization of the above idea is to postulate the commutation relations among spacetime coordinates

$$[x^i, x^j] = i\theta^{ij}, \tag{1.1}$$

with a constant parameter θ which is an antisymmetric tensor. However, there are two questions which should be answered: 1) Could the commutation relations (1.1) be indeed reproduced by a quantum theory including gravity? 2) How to study gravity fluctuations of on the noncommutative backgrounds, i.e., to construct a noncommutative theory of quantum gravity?

To answer these questions, a consistent quantum theory including gravity should be *a priori* known. String theory, at least perturbatively, may be such a candidate. Unfortunately, perturbative string theory itself is failed to answer the first question since the background has to be input in advance. The current terminology "noncommutative geometry in string theory" often states as noncommutativity of world-volume coordinates of D-branes[2, 3, 4, 5] or equivalently of coordinates of end-points of open strings[6]. It induces various noncommutative quantum field theories without gravity on world-volume of D-brane. These theories can be viewed as a sort of effective description on open strings ending on D-branes in a background with large constant NS field along world-volume of D-branes[7]. This background does not directly change anything of closed strings as well as of gravity at lower energy. Instead, the amplitudes involving closed strings are changed

through coupling between closed string and open string[8, 9, 10]. These studies, therefore, can not answer the second questions too.

Although there are many difficulties, we may ask an alternative question: Is it possible to construct the perturbative string theory on the background with commutation relations (1.1)? In this present paper we shall contribute a small step along this direction, that is to study the light-cone quantization of bosonic strings on the flat background with commutation relations (1.1) and extract energy spectrum of string oscillation. The progress is almost the same as standard progress for strings on flat commutative background, except that the additive noncommutativity of world-sheet scalar fields changes commutators among raising and lower operators. Consequently the energy spectrum of strings on the commutative background is modified.

The directly generalization of Eq. (1.1) to the "equal time" canonical commutation relations of world-sheet scalar fields are as follows,

$$[X^i(\tau,\sigma), X^j(\tau,\sigma')] = \begin{cases} i\alpha'\theta^{ij}, & \text{if} \quad \sigma = \sigma', \\ 0, & \text{if} \quad \sigma \neq \sigma', \end{cases} \tag{1.2}$$

with τ, σ world-sheet coordinates and α' the Regge slope. In equation (1.2) we rescale the antisymmetric tensor θ so that it is dimensionless now. When $l_{nc} \sim l_s$ with l_{nc} the noncommutative length and $l_s = \sqrt{\alpha'}$ the string length, however, there is an extra complication: The commutation relations (1.1) implies an uncertainty region order l_{nc}. The distance of any two points inside this region can not be measured accurately. On the other hand, $l_s \sim l_{nc}$ indicates that the whole of string is included into such an uncertainty region, hence the positions of any two points on strings are in general noncommutative,

$$[X^i(\sigma), X^j(\sigma')] = i\alpha'\theta^{ij}\mathcal{F}(\sigma - \sigma'). \tag{1.3}$$

The commutation relations (1.3) yields large ambiguities to build noncommtative models of strings. However, we may expect that Eq. (1.2) is a sort of limit of (1.3). It imposes a strong constraint to function $\mathcal{F}(\sigma - \sigma')$. Hereafter we shall call the models with commutation relations (1.2) as "rigorous noncommutative model", and those with relations (1.3) as "fuzzy noncommutative model". These two kinds of models, as shown in our studies, yield very different physics.

In section 2 of this paper, we study the light-cone quantization of the rigorous noncommutative model. While a fuzzy noncommutative model is investigated in section 3. It shares some basic properties of any fuzzy models. Discussions and conclusions are included in section 4.

2. Rigorous noncommutative model

The commutation relations (1.2) and (1.3) do not modify classical action of bosonic strings on D-dimensional flat spacetime,

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-g} g^{ab}\partial_a X^\mu \partial_b X^\mu, \tag{2.1}$$

as well as classical equations of motion. Consequently the conformal symmetry on world-sheet is kept at classical level at least. Since quantum theories with noncommutativity in time may have problem with unitarity[11], we will consider only spatial noncommutativity, i.e., $\theta^{0i} = 0,\ (i = 1, 2, ..., D-1)$. Then we can take light-cone gauge that is the same as one on commutative background. The mode expansions of transverse world-sheet scalar fields are also the same, i.e.,

$$X^i = x^i + \frac{p^i}{p^+}\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left\{\frac{\alpha_n^i}{n}e^{-in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^i}{n}e^{-in(\tau-\sigma)}\right\}, \qquad \sigma \in [0, 2\pi], \quad (2.2)$$

for closed strings and

$$X^i = x^i + \frac{p^i}{p^+}\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\alpha_n^i}{n}e^{-in\tau}\cos n\sigma, \qquad \sigma \in [0, \pi] \quad (2.3)$$

for open strings. It is not hard to check that the following canonical commutators

$$\begin{aligned} [\alpha_n^i, \alpha_m^j] &= [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta_{m+n,0}\left\{\delta^{ij} + \frac{2i}{\pi^2}\theta^{ij}\lim_{L\to\infty}\sum_{l=1}^{L}\frac{n}{n^2-(l-1/2)^2}\right\}, \\ [x^i, p^j] &= i\delta^{ij}, \qquad\qquad [x^i, x^j] = [p^i, p_j] = 0 \end{aligned} \quad (2.4)$$

yield the "equal time" canonical commutation relations (with Π^i canonical momenta)

$$\begin{aligned} [X^i(\sigma), \Pi^j(\sigma')] &= i\delta^{ij}\delta(\sigma-\sigma'), \\ [X^i(\sigma), X^j(\sigma')] &= \begin{cases} i\alpha'\theta^{ij}, & \text{if} \quad \sigma = \sigma', \\ 0, & \text{if} \quad \sigma \neq \sigma' \end{cases} \end{aligned} \quad (2.5)$$

for both of closed and open strings. It is just that we wanted. While commutation relations among Π^i are divergent

$$[\Pi^i(\sigma), \Pi^j(\sigma')] \ \propto \ iL\theta^{ij}\delta(\sigma-\sigma') + ... \quad (2.6)$$

In equation (2.4) we defined the infinite summation as a limit since the summation

$$\sum_{l=1}^{\infty}\frac{n}{n^2-(l-1/2)^2}$$

is not well-defined for $n \to \infty$.

The ground states for oscillation of closed strings, $|0, 0; k\rangle$ can again be defined to be annihilated by the lowering operators $\alpha_n^i,\ (n > 0)$ and to be eigenstates of the center-of-mass momenta. Furthermore, the general eigenstates of Hamiltonian,

$$H = \frac{1}{2p^+}p^ip^i + \frac{1}{p^+\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i) + \frac{2-D}{12}, \quad (2.7)$$

are constructed by acting on $|0,0;k\rangle$ with the raising operators,

$$|N,\tilde{N};k\rangle = \prod_{i=2}^{D-1}\prod_{n=1}^{\infty} \frac{(\beta^i_{-n})^{N_{in}}(\tilde{\beta}^i_{-n})^{\tilde{N}_{in}}}{(\lambda_{in}^{N_{in}+\tilde{N}_{in}} N_{in}!\tilde{N}_{in}!)^{1/2}}|0,0;k\rangle, \tag{2.8}$$

where

$$\begin{cases} \beta_{-n} = U\alpha_{-n}, \\ \beta_n^T = \alpha_n^T U^\dagger, \end{cases} \qquad \begin{cases} \tilde{\beta}_{-n} = U\tilde{\alpha}_{-n}, \\ \tilde{\beta}_n^T = \tilde{\alpha}_n^T U^\dagger, \end{cases} \qquad n > 0,$$

$$\lambda_{in} = n + \frac{2\nu_i}{\pi^2}\lim_{L\to\infty}\sum_{l=1}^{L}\frac{n^2}{n^2-(l-1/2)^2}. \tag{2.9}$$

with U unitary matrix which diagonalizes anti-symmetrical matrix θ and ν_i the eigenvalues of $i\theta$ (so that all of ν_i are real).

Finally we obtain the energy spectrum,

$$E^2 = k^i k^i + \frac{2}{\alpha'}\left\{\sum_{n=1}^{\infty}\lambda_{in}(N_{in}+\tilde{N}_{in}) + \frac{2-D}{12}\right\}, \tag{2.10}$$

with level match condition

$$\sum_{n=1}^{\infty}\lambda_{in}(N_{in}-\tilde{N}_{in}) = 0. \tag{2.11}$$

Since

$$\sum_{l=1}^{L}\frac{n}{n^2-(l-1/2)^2} = \sum_{l=1}^{2n}\frac{1}{2(l+L-n)-1}, \tag{2.12}$$

at the limit $L\to\infty$ energy spectra of finite n modes do not receive modifications from noncommutative effect. In other words, the dimensions of spacetime is still required 26 and the massless states[1] are the same as those in commutative case. For those modes with $n = qL \to \infty$, $q \ll 1$, however, their energy spectra are modified due to $\lambda_{in} \simeq n(1-4q\nu_i/\pi^2)$. It reflects a fact that noncommutative effects can only be detected by extreme high energy modes.

The above processes can be applied to open strings and the same conclusions are obtained.

3. Fuzzy noncommutative model

The rigorous model studied in previous section has an obvious inconsistency at extreme short distance: The summation (2.12) is logarithmically divergent for $n/L \simeq 1,\ L\to\infty$.

[1]The noncommutativity of spacetime violates the Lorentz symmetry, so that in general we can not define what is "mass" of a particle. In this paper we will usually refer "energy spectrum" instead of "mass spectrum". At low energy, however, the Lorentz invariance is good symmetry so that the massless states can be defined as usual.

Since antisymmetric matrix $i\theta$ always has negative eigenvalue, the modes with extreme short wave length (or sufficient large n) have ill-defined energy $E^2 \to -\infty$. It supports the argument in Introduction that the commutation relations (1.2) should be replaced by (1.3) when string scale is order to Plank scale. In this section we will consider a simple fuzzy model in which L in commutation relations (2.4) is set to be finite integer. Hence the rigorous model is a limit of this fuzzy model. Although this model is very special, it shares some basic features of this type of models.

For finite L, the summation (2.12) is also finite. So that the noncommutativity yields a finite modification to whole string spectrum except for mode with $n = 0$. In particular, this modification completely breaks the spatial rotation $SO(D-2)$ symmetry of massless states ($n = 1$ modes) on original commutative background. For general choice of antisymmetric matrix θ, there are no any massless states in this model. In order for this model to describe the physics of our world, the eigenvalues of θ must be degenerate. It is particular interest to consider that the eigenvalues of θ is double degenerated, which corresponds that $D \times D$ matrix θ can be reduced to a diagonal block form as follows

$$\theta = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & A & \\ & & & A \end{pmatrix}. \tag{3.1}$$

with A antisymmetric $(D/2-1) \times (D/2-1)$ matrix. Consequently dimensions of spacetime must be even and coordinates in this representation may be divided into three group: $D = (1 + 1) + (D/2 - 1) + (D/2 - 1)$. Coordinates in different groups are commutative each other, while those in one of latter two groups are noncommutative.

Let $\nu_{\min} < 0$ be minimal eigenvalue of $i\theta$. In order that there are massless states in $n = 1$ modes, we require

$$1 + \frac{2\nu_{\min}}{\pi^2} \sum_{l=1}^{L} \frac{1}{1 - (l - 1/2)^2} = \frac{D-2}{24}. \tag{3.2}$$

Therefore, the critical dimension in this model is modified to $D \leq 24$ by noncommutative effect since $\nu_{\min}$ is order one when string scale is order noncommutative scale. Furthermore, to avoid to appearance of new states with negative energy square, we should impose the condition $\lambda_{in} \geq 0$. Since maximal value of summation (2.12) corresponds to $n = L$, one has

$$1 + \frac{2\nu_{\min}}{\pi^2} \sum_{l=1}^{L} \frac{L}{L^2 - (l - 1/2)^2} = 1 + \frac{\nu_{\min}}{\pi^2} \sum_{l=1}^{2L} \frac{1}{l - 1/2} > 0. \tag{3.3}$$

Equation (3.3) together with (3.2) yield

$$\frac{8L}{4L^2 - 1} + \frac{D-26}{24} \sum_{l=1}^{2L} \frac{1}{l - 1/2} > 0. \tag{3.4}$$

L	critical dimension D
1	$6,\ 8, ..., 24$
2	$20,\ 22,\ 24$
3	$22,\ 24$
$4,\ 5$	24
≥ 6	no consistent choice

Table 1: The critical dimensions for fuzzy noncommutative string model admitted by diverse values of L.

The above equation impose a constraint between L and critical dimension D. In table 1 we list the critical dimensions admitted by diverse values of L. We can see that only few choices to L are allowed.

Using commutators (2.4), the function $\mathcal{F}(\sigma-\sigma')$ defined in commutation relations (1.3) can be obtained

$$\mathcal{F}(\sigma-\sigma') = \frac{2}{\pi}\sum_{l=1}^{L}\left\{\frac{1}{(l-1/2)^2} - \frac{\pi}{l-1/2}\sin{(l-\frac{1}{2})(\sigma-\sigma')}\right\}. \tag{3.5}$$

Physically, we expect that accurate measure of position of a point is much more difficult than accurate measure of distance between two points even though these two points are both included in the same uncertainly region. In other words, we expect that the fuzzy model should satisfy

$$q = \frac{1}{2\pi\mathcal{F}(0)}\int_0^{2\pi} d\sigma|\mathcal{F}(\sigma)| \ll 1. \tag{3.6}$$

For our special model, the above condition implies that L should be chosen as large as possible. Indeed, direct calculations show that $q = 0.42$ for $L = 1$ and $q = 0.12$ for $L = 5$. So that $L = 5$ is a better choice which requires $D = 24$.

No matter how many spacetime dimensions are admitted, at low energy there are only four massless states generated by oscillation of closed strings, and two massless states generated by open strings. They can be interpreted as a graviton, a gauge boson and two scalars in four dimensions. In other words, the low-energy spectrum of fuzzy model may contains degrees of freedom of our $(3+1)$-dimension physics only and exhibits a $(3+1)$-dimension Lorentz invariance. This is a basic properties of the fuzzy models. Those models, therefore, may be regards as an alternative compactification of string theory.

4. Summary and discussion

To conclude, we investigated bosonic strings in a flat but noncommutative spacetime. The string spectrum is obtained in terms of light-cone quantization. The low-energy physics predicted by rigorous model is almost the same as usual bosonic string in commutative background. However, the rigorous model is inconsistent when string scale is order non-commutative scale. We further proposed a fuzzy model which may avoid the inconsistency

of the rigorous model at very short distance. The critical dimensions and low-energy degrees of freedom in the fuzzy model, however, are reduced by noncommutative effect. In particular, it is possible to construct a model in which massless spectrum only contains degrees of freedom of our four-dimensional world although many extra dimensions are presented. This is typical nonlocal effect: Physics at short distance produces observable effect at long distance.

It should be asked what is physical meaning to consider a string theory in noncommutative background: String theory itself is nonlocal. While any perturbative quantum theory on noncommutative spacetime must exhibit nonlocal effects too. Then is there problem of double counting? If we believe that spacetime must change its nature at very short distance by supplement of commutation relations such as (1.1) and non-perturbation definition of string theory (or M-theory) is a correct quantum theory to describe our world with gravity, we should expect that the spacetime with commutation relations (1.1) is a non-perturbative vacuum of string theory. Hence it does make sense to consider the perturbative excitations of strings on this vacuum.

The further progresses may focus on CFT approach and supersymmetric generalization of these noncommutative model. Unfortunately, there is a technical trouble to construct CFT approach: The operator equation derived from

$$0 = \int [dX] \frac{\delta}{\delta X_\mu(z,\bar{z})} \left[e^{-S} X^\nu(z',\bar{z}') \right] \tag{4.1}$$

is highly nonlinear and hard to solve. It prevents us from obtaining the operator product expansion. It is expected to overcome this difficulty in future studies.

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References

[1] B. DeWitt, *Gravitation*, edited by L. Witten, (1962) 266-381.

[2] M. Li and T. Yoneya, *D-Particle Dynamics and The Space-Time Uncertainty Relation*, Phys. Rev. Lett. **78** (1997) 1219, [hep-th/9611072].

[3] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative Geometry and Matrix Theory: Compactification on Tori*, JHEP **02** (1998) 003, [hep-th/9711162].

[4] M.R. Douglas and C. Hull, *D-branes and the Noncommutative Torus*, JHEP **02** (1998) 008, [hep-th/9711165].

[5] C. Hofman and E. Verlinde, *U-Duality of Born-Infeld on the Noncommutative Two-Torus*, JHEP **12** (1998) 010, [hep-th/9810116].

[6] C.-S. Chu and P.-M. Ho, *Noncommutative Open String and D-brane*, Nucl. Phys. **B550** (1999) 151, [hep-th/9812219].

[7] N. Seiberg and E. Witten, *String Theory and Noncommutative Geometry*, JHEP **09** (1999) 032, [hep-th/9908142].

[8] S.R. Das and S.P. Trivedi, *Supergravity couplings to Noncommutative Branes, Open Wilson Lines and Generalised Star Products*, JHEP **02** (2001) 046, [hep-th/0011131].

[9] H. Liu and J. Michelson, *Supergravity Couplings of Noncommutative D-branes*, Nucl. Phys. **B615** (2001) 169, [hep-th/0101016].

[10] Y. Okawa and H. Ooguri, *How Noncommutative Gauge Theories Couple to Gravity*, Nucl. Phys. **B599** (2001) 55, [hep-th/0012218].

[11] J. Gomis and T. Mehen, *Space-Time Noncommutative Field Theories And Unitarity*, Nucl. Phys. **B591** (2000) 265, [hep-th/0005129];
L. Alvarez-Gaume, J.L.F. Barbon and R. Zwicky, *Remarks on Time-Space Noncommutative Field Theories*, JHEP **05** (2001) 057, [hep-th/0103069];
C.-S. Chu, J. Lukierski and W.J. Zakrzewski, *Hermitian Analyticity, IR/UV Mixing and Unitarity of Noncommutative Field Theories*, Nucl. Phys. **B632** (2002) 219, [hep-th/0201144].